example continued from last day....

$$K_R = \frac{1}{2}\omega_1^T I_1 \omega_1 + \frac{1}{2}\omega_2^T I_2 \omega_2$$

$$= \frac{1}{2}\dot{q}\left[J_{\omega_1}^T \, {}^0_1R \, I_1 \, {}^0_1R^T J_{\omega_1} + J_{\omega_2}^T \, {}^0_2R \, I_2 \, {}^0_2R^T J_{\omega_2}\right]\dot{q}$$

$$I_1 = \begin{bmatrix} x & & \\ & x & \\ & & I_{zz1} \end{bmatrix}$$

$$I_2 = \begin{bmatrix} x & & \\ & x & \\ & & I_{zz2} \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$K_R = \frac{1}{2} I_{zz1}\dot{q}_1^2 + \frac{1}{2} I_{zz2}(\dot{q}_1 + \dot{q}_2)^2$$

$$= \frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix} I_{zz1} + I_{zz2} & I_{zz2} \\ I_{zz2} & I_{zz2} \end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}$$

$$K = K_T + K_R = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

$$D(q) = m_1 J_{vc_1}^T J_{vc_1} + m_2 J_{vc_2}^T J_{vc_2}$$

$$+ \begin{bmatrix} I_{zz1} + I_{zz2} & I_{zz2} \\ I_{zz2} & I_{zz2} \end{bmatrix}$$

$$= \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

note: $d_{12} = d_{21}$

$$d_{11} = m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos q_2) + I_{zz1} + I_{zz2}$$

$$d_{21} = d_{12} = m_2(l_{c2}^2 + l_1 l_{c2} \cos q_2) + I_{zz2}$$

$$d_{22} = m_2 l_{c2}^2 + I_{zz2}$$

Christoffel Symbols.

$$c_{111} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0$$

$$c_{221} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h$$

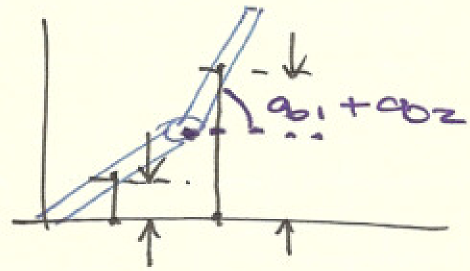
$$c_{121} = c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = m_2 l_1 l_{c2} \sin q_2 = h$$

$$c_{112} = \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h$$

$$c_{122} = c_{212} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0$$

$$c_{222} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0$$

Potential Energy.



$$V_1 = m_1 g l_{c1} \sin q_1$$

$$V_2 = m_2 g (l_1 \sin q_1 + l_{c2} \sin(q_1+q_2))$$

$$V = V_1 + V_2$$

$$\frac{\partial V}{\partial q_1} = m_1 g l_{c1} \cos q_1 + m_2 g l_1 \cos q_1 + m_2 g l_{c2} \cos(q_1+q_2)$$

$$\frac{\partial V}{\partial q_2} = m_2 l_{c2} \cos(q_1+q_2)$$

Equations of motion.

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_1\dot{q}_2$$

$$+ C_{221}\dot{q}_2^2 + \frac{\partial V}{\partial q_1} = \tau_1$$

$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + C_{112}\dot{q}_1^2 + \frac{\partial V}{\partial q_2} = \tau_2$$

These are the equations of motion, they are the final diagonal model and can be expressed as.

$$C(q,\dot{q}) = \begin{bmatrix} h\dot{q}_2 & h(\dot{q}_1 + \dot{q}_2) \\ -h\dot{q}_1 & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} \frac{\partial V}{\partial q_1} \\ \frac{\partial V}{\partial q_2} \end{bmatrix}$$

$$D(q) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad \ddot{q} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} \quad \tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Euler - Newton Method. (formulation)

The Newton-Euler formulation is based on the following facts.

1. Every action has an opposite and equal reaction
2. The rate of change of the linear momentum equals to the total force applied to the body.
3. The rate of change of the angular momentum equals to the total tourque applied to the body.

Applying the second fact to the linear motion of a body gives

velocity of COM

$$\frac{d(mv)}{dt} = f.$$

$$m \frac{dv}{dt} = f$$

mass of body.

$$f = ma.$$

sum of external forces.

If the mass is constant.

Applying the 3rd fact to the angular motion of a body gives.

angular velocity.

$$\frac{d(I_o w_o)}{dt} = \tau_o$$

sum of external tourque

inertia of the body about an inertial frame whose origin is at the center of mass.

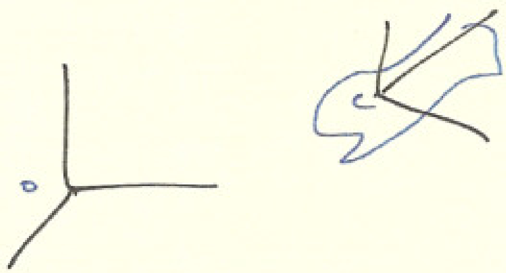
note: I_o ~~is the~~ can be time varying which makes this hard to calculate.

One possible way to overcome this difficulty is to write the angular momentum in terms of the frame that is rigidly attached to the body.

I : is expressed in the frame rigidly attached to the body.

$$I = {}^{T}R\, I_o$$

$$w = {}^{T}R\, w_o$$



$$\therefore \quad \frac{d(I_o w_o)}{dt} = \tau_o$$

$$\frac{d}{dt}(R I R w) = R^T \tau$$

I is now costant

τ is ~~expressed~~ in the body of the fixed frame

we end up with.

$$\left(\frac{dR}{dt}\right) I R w + R I \frac{d(Rw)}{dt} = R^T \tau .$$

$$\boxed{I \dot{\omega} = \underbrace{- \omega \times (I \omega)}_{\text{caused due to 3D space.}} + \tau} \quad (1)$$

note: we use $\dot{R} R = S(\omega_o)$

Newton Euler formulation of N-DOF rigid robot manipulator.

We first chose frames $0, 1 \ldots n$ where frame 0 is an inertial frame and frame i is rigidly attached to ~~frame~~ link i.

$a_{c,i}$: acceleration of COM of link i

$a_{e,i}$: " " end of link i.

w_i : angular velocity of frame i w.r.t. frame 0.

α_i: angular accelleration of frame i w.r.t frame 0.

g_i: the accelleration due to gravity expressed in frame i

f_i: the force extended by link i-1 on link i.

τ_i: the tourque exerted by link i-1 on link i

${}^{i}_{i+1}R$: Rotation matrix from frame i+1 to frame i

m_i: mass of link i.

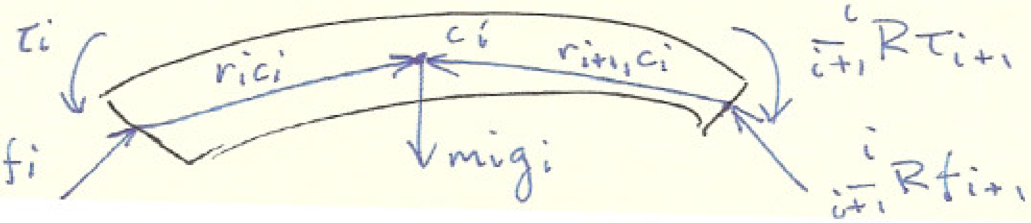
I_i: inertia matrix of link i about a frame parrellel to frame i whose origin is at the COM of link i.

$r_{i,ci}$: the vector from joint i to the COM of link i.

$r_{i+1,ci}$: the vector from joint i+1 to the center of mass of link i.

$r_{i,i+1}$: the vector from joint i to joint i+1.

Consider the free body diagram of link i.



f_i is the force applied to link $i-1$ to link i by the law of action and reaction, link $i+1$ applies a force of $-f_{i+1}$ to link i but this vector is expressed in frame $i+1$ according to convention